Int. J. Heat Mass Transfer. Vol. 22, pp. 499–504
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0017-9310/79/0401-0499 $02.00/0

TRANSIENT BEHAVIOR OF NATURAL CIRCULATION LOOPS: TWO VERTICAL BRANCHES WITH POINT HEAT SOURCE AND SINK

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(*Received* 1 *June* 1978 *and in revised form* 31 *August* 1978)

Abstract—A theoretical method is presented for the evaluation of the transient behavior of free convection loops. The method is applied to a loop consisting of two vertical branches with a point heat source and sink. The system is represented by a one dimensional model, with the only space coordinate running along the loop. Integral forms of the momentum and energy equations are derived and solved to yield the flow rate and temperatures as functions of the time. It is found that this approximate method cannot reconstruct the stability characteristics of the exact steady state solution.

NOMENCLATURE

A, area of cross section;
c, specific heat;
g, acceleration of gravity;
h, heat-transfer coefficient per unit of length;
H, penetration height;
K, defined by $h/\rho_{\text{ref}}cA$;
L, length;
Q, volumetric flow rate;
s, coordinate along the loop;
T, temperature;
t, time;
t^*, time interval for first stage.

Greek symbols

α, dimensionless parameter, equation (3);
β, thermal expansion coefficient;
ε, dimensionless parameter, equation (3);
λ_1,λ_2, quantity defined in equation (13c);
σ, stability parameter, equation (A.4);
ρ_{ref}, reference density.

Subscripts

0, location $s = 0$;
1, location $s = 1$;
i, initial value;
m, mean value.

Superscripts

$\bar{\ }$, steady state;
$'$, deviation from steady state;
$\hat{\ }$, stability quantity, equation (A.4).

1. INTRODUCTION

MOST of the existing studies on natural circulation loops are concerned with stability of steady-state motion. Keller [1] and Welander [2] investigated analytically a point heat source, point heat sink loop with two vertical branches. They presented an explanation for instabilities occuring in some laminar flow situations by considering the phase shift between the flow rate and buoyancy forces. Creveling *et al.* [3] treated a toroidal loop and found, experimentally and theoretically, instabilities in the transition zone between the laminar and the turbulent flow regimes. Zvirin *et al.* [4, 5] studied the stability characteristics of the thermosyphonic solar water heater and showed that this system can become unstable at high energy utilizations.

Ong [6] suggested a numerical method for the evaluation of the transient behavior of the solar water heater. His solution, however, is based on some approximations which are not generally valid, e.g. negligible inertia of the fluid, linear temperature distributions and uniform mean temperatures in all system components.

This work presents a method for the study of transient phenomena in a free convection loop. The system considered is comprised of two vertical branches with a point heat source and sink (cf. Fig. 1). Following the earlier studies some basic assumptions are made. The model is one dimensional with the space coordinate s being taken along the circulation loop. The Boussinesq approximation is used, whereby the density ρ is taken to be constant in the governing continuity, momentum and energy equations except in the body force term, where $\rho = \rho_{\text{ref}}[1-\beta(T-T_{\text{ref}})]$. All the fluid properties and heat-transfer coefficients are assumed to be constant. With these assumptions the flow rate Q is uniform along the loop at any time t. The analysis is for laminar flow, where the friction force depends linearly on the flow rate.

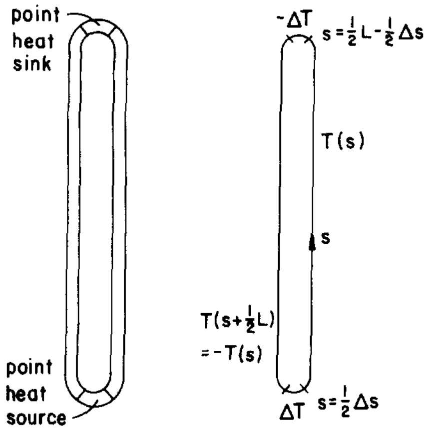


FIG. 1. The loop of two vertical branches, a point heat source and a point heat sink.

To determine the transient behavior of the system, integral forms of the momentum and energy equations have been derived. The governing equations are thus reduced to coupled ordinary non-linear differential equations in time. Linear temperature distributions are assumed and the equations are then solved numerically. As expected, the results for large times tend asymptotically to the steady state values reported by Welander [2]. However, the solution obtained here for the transient behavior does not indicate any instability, even for the range of parameters where the steady state motion was found by Welander [2] to be unstable. It is noted that stability strongly depends on the shape of the temperature distribution. Thus, while the linear temperature distributions can describe the steady state motion as established by Zvirin *et al.* [4], such distributions cannot account for the instabilities in the loop considered here.

2. ANALYSIS

The loop considered here is shown in Fig. 1. Welander [2] studied the steady motion and stability of this loop. The point heat source and sink are represented by constant wall temperatures ΔT and $-\Delta T$ that act over small portions of the loop (Δs) at the bottom and top (respectively). Following Welander [2], the limiting case $\Delta s \to 0$ is considered with an overall infinite heat-transfer coefficient per unit length† there, $h \to \infty$, such that the heat flux remains finite (point source and point sink). Note that the system is antisymmetric so that only one branch has to be considered.

We write the governing equations in the following non-dimensional form, cf. Welander [2]:

$$\frac{dQ}{dt} + \varepsilon Q = \alpha \int_0^1 T\, ds \qquad (1)$$

$$\frac{\partial T}{\partial t} + Q\frac{\partial T}{\partial s} = 0 \qquad 0 < s < 1, \qquad (2)$$

†Note that the overall heat-transfer coefficient multiplied by the perimeter is denoted by h.

where the dimensionless parameters ε and α are defined by:

$$\alpha = \frac{g\beta \Delta T L}{2(K\Delta s)^2}, \quad \varepsilon = \frac{RL}{2K\Delta s}. \qquad (3)$$

In these dimensionless equations, length is scaled by $L/2$, time by $L/2K\Delta s$, flow rate by $KA\Delta s$ and temperature by ΔT. R is a frictional coefficient such that $\rho_{ref}LRQ$ is the total friction force and $K = h/\rho_{ref}cA$, where c is the specific heat of the fluid and A is the cross-sectional area. The boundary conditions for the temperature are obtained from a balance on the heat source or sink:

$$T_0 + T_1 = (1 + T_1)(1 - e^{-1/Q}) \text{ for } Q > 0 \qquad (4a)$$

$$T_0 + T_1 = (-1 + T_0)(1 - e^{-1/Q}) \text{ for } Q < 0 \qquad (4b)$$

where the subscripts 0 and 1 denote temperatures at $s = 0,1$ (respectively). The solution of equations (1) and (2) requires initial values for T and Q. We consider the case where the heat source and sink are applied initially to a stationary loop of uniform temperature $T = 0$. Since there exists no flow at $t = 0$, the fluid at the source is heated such that its temperature equals that of the wall. The initial conditions are, therefore:

$$Q = 0, \; T_0 = 1, \; T(s) = 0 \qquad 0 < s < 1. \qquad (5)$$

It is noted that due to symmetry, a metastable state can exist with no flow, hot fluid at the bottom and cold fluid at the top. The onset of the flow is therefore involved with an instability. However, the analysis of this initiation of the flow is beyond the scope of the present study and we assume that the flow starts immediately due to some small non-symmetry.

The solution is separated into two stages as shown in Fig. 2. During the first stage a linear temperature distribution is taken from the bottom ($s = 0$) to a penetration height $H(t)$, and $T = 0$ for $H \leqslant s < 1$. The penetration height increases with time until $H = 1$ at $t = t^*$ and the second stage then begins. For $t > t^*$ a linear temperature distribution is assumed over the whole range $0 < s < 1$.

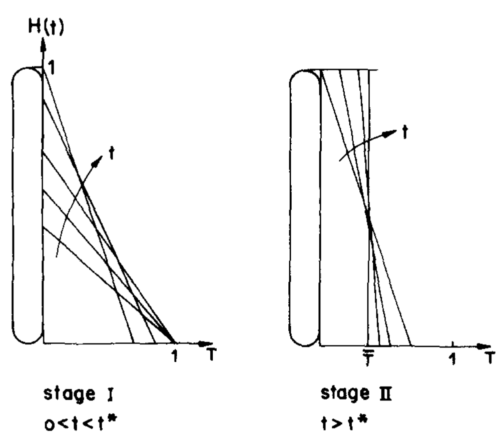


FIG. 2. The model for variation of the temperature distribution with time.

For the first stage, integration of the energy equation (2) yields:

$$\frac{d}{dt}\int_0^H T\,ds - QT_0 = 0 \qquad 0 < t < t^*, \tag{6}$$

where the condition $T_H = 0$ was used. For a linear temperature distribution equation (6) reduces to:

$$\frac{1}{2}\frac{d}{dt}(HT_0) = QT_0 \qquad 0 < t < t^*, \tag{7}$$

and the momentum equation (1) becomes

$$\frac{dQ}{dt} + \varepsilon Q = \tfrac{1}{2}\alpha T_0 H, \tag{8}$$

while the condition (4a) takes the form:

$$T_0 = 1 - e^{-1/Q} \qquad 0 < t < t^* \tag{9}$$

(Q is taken to be positive). Equations (7)–(9) constitute a set of three coupled ordinary differential equations for the three variables H, T_0 and Q. The initial conditions are:

$$Q = 0,\ T_0 = 1,\ H = H_i \text{ at } t = 0, \tag{10}$$

where H_i is a small value representing the initial non-symmetry of the heat source. The set (7)–(9) can be reduced to the following single equation by differentiating equation (8) with respect to time and introducing equations (7) and (9):

$$\frac{d^2Q}{dt^2} + \varepsilon\frac{dQ}{dt} = \alpha Q(1 - e^{-1/Q}) \quad 0 < t < t^* \tag{11}$$

with the initial conditions:

$$Q = 0,\ \frac{dQ}{dt} = \tfrac{1}{2}\alpha H_i \text{ at } t = 0. \tag{12}$$

The last condition results from (8) and (10). Equation (11) is non-linear and generally must be solved numerically. However, for small values of α/ε, an analytical solution can be obtained. For this case at steady state the flow rate is small (of order α/ε) and the temperature T along the right branch is 1. During the first stage Q is even smaller and the RHS of equation (11) reduces to αQ. The solution is then given by:

$$Q = \frac{\alpha H_i}{2(\varepsilon^2 + 4\alpha)^{1/2}}(e^{\lambda_1 t} - e^{\lambda_2 t}) \tag{13a}$$

$$H = \frac{H_i}{(\varepsilon^2 + 4\alpha)^{1/2}} \times [(\lambda_1 + \varepsilon)e^{\lambda_1 t} - (\lambda_2 + \varepsilon)e^{\lambda_2 t}] \tag{13b}$$

with

$$\lambda_{1,2} = -\varepsilon/2 \mp (\varepsilon^2 + 4\alpha)^{1/2}. \tag{13c}$$

It is seen that the solution, and hence also the time $t = t^*$ when H reaches the top ($H = 1$), depends on the initial value H_i. The whole solution tends to zero as $H_i \to 0$.

The first stage terminates at $t = t^*$ when $H = 1$ and the second stage then begins. The integral of the energy equation (2), is:

$$\frac{d}{dt}\int_0^1 T\,ds + Q(T_1 - T_0) = 0 \qquad t > t^*. \tag{14}$$

The integral $\int_0^1 T\,ds$ equals $T_m = (T_0 + T_1)/2$. Making use of condition (4a), the temperatures T_0 and T_1 are expressed in terms of T_m as follows:

$$T_0 = 1 - \frac{2T_m e^{-1/Q}}{1 - e^{-1/Q}},\quad T_1 = \frac{2T_m}{1 - e^{-1/Q}} - 1 \tag{15}$$

and equation (14) then takes the form:

$$\frac{dT_m}{dt} = 2Q\left[1 - \frac{(1 + e^{-1/Q})}{1 - e^{-1/Q}}T_m\right] \quad t > t^*. \tag{16}$$

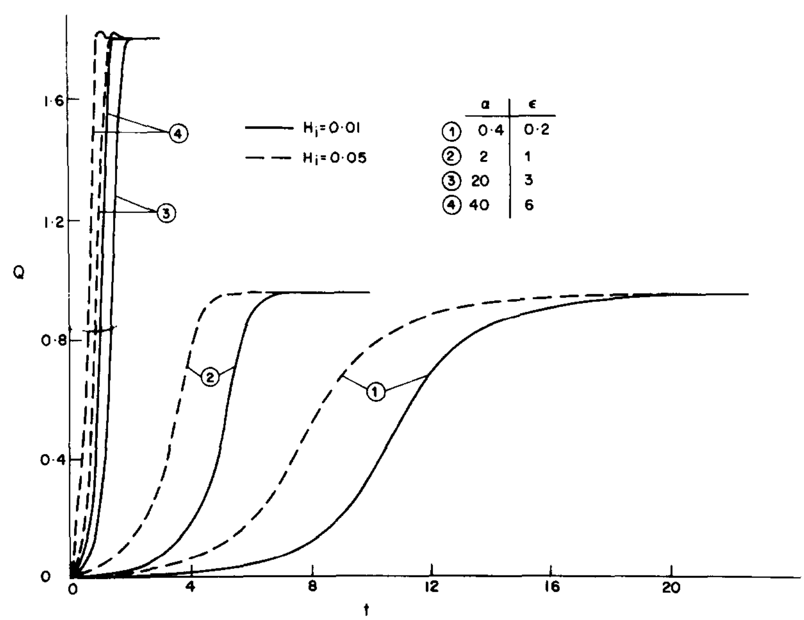
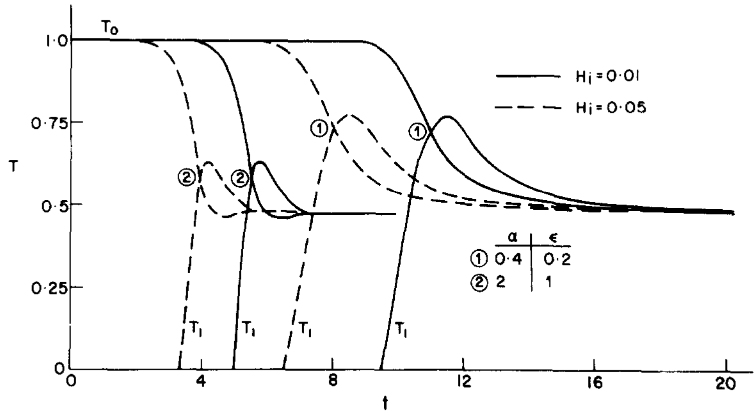


FIG. 3. Flow rate as a function of time for various system parameters α and ε.



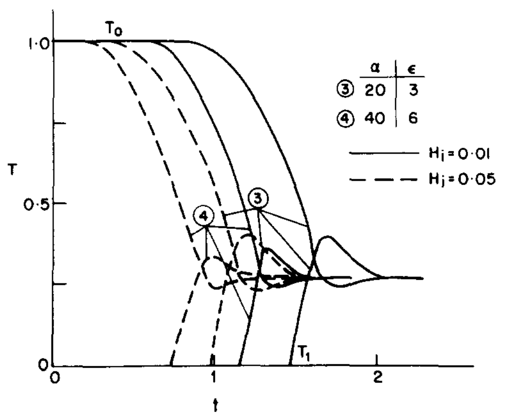


FIG. 4. Temperatures at the bottom and top of the loop as functions of time for various system parameters α and ε.

The momentum equation (1) is now written as:

$$\frac{dQ}{dt} = -\varepsilon Q + \alpha T_m \qquad t > t^*. \tag{17}$$

Equations (16) and (17) constitute a set of two non-linear coupled ordinary differential equations for the two variables T_m and Q. The initial conditions are:

$$Q = Q_{\text{stage I}},\ T_m = \tfrac{1}{2}T_0 \ \text{ at } \ t = t^*. \tag{18}$$

It is also noted that for this stage there exists no analytical solution even for the special case of small α/ε.

3. RESULTS AND DISCUSSION

The solution procedure is the following: (11) with the initial conditions (12) for stage I is solved numerically for $Q(t)$. The maximal temperature $T_0(t)$ is then determined from (9) and the height $H(t)$ from (8). Note that dQ/dt is available for every time step as a part of the numerical solution. This solution continues until $H = 1$. At this time, $t = t^*$, stage II begins and the behavior of the system is then obtained from a numerical integration of (16) and (17) subject to the conditions (18). This solution yields $Q(t)$ and $T_m(t)$, and (15) are then used to compute T_0 and T_1. The calculation continues until steady state is reached.

In the present study the Runge–Kutta method was used. Two initial values of H_i were chosen, 0.05 and 0.01, along with various values of α and ε, which were also studied by Welander [2]. The results for $Q(t)$ are shown in Fig. 3, $T_0(t)$ and $T_1(t)$ are presented in Fig. 4 and $H(t)$ in Fig. 5. All the correct steady state values are reached asymptotically at rates depending on α, ε and H_i. It is seen that all the curves are smooth and do not show any instabilities.

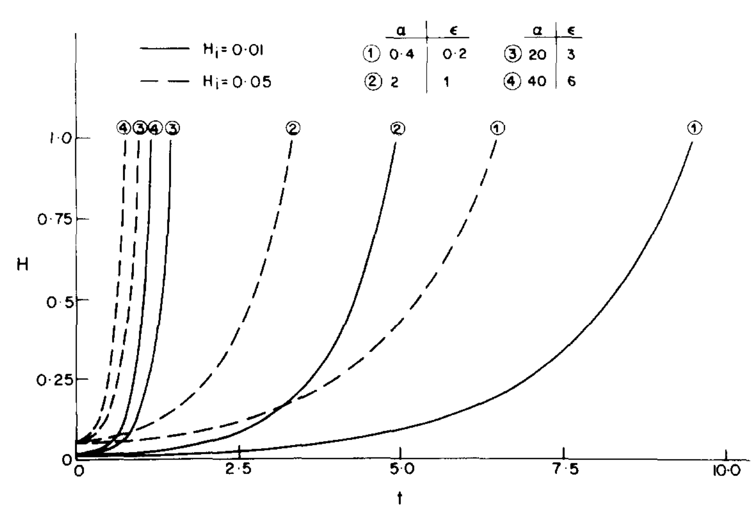


FIG. 5. The growth of the "heated region" of the loop vs time for various system parameters α and ε.

According to Welander [2] the steady state of $\alpha = 20$, $\varepsilon = 3$ is close to neutral oscillation and that of $\alpha = 40$, $\varepsilon = 6$ is unstable. For these cases he obtained strong oscillations on the basis of a finite difference numerical solution of the original partial differential equations. In the present procedure, only very small oscillations appeared at steady state and these are not discernible in Figs. 3 and 4.

It is emphasized that the approximate equations (15)–(17) yield the exact steady state solution; the temperature is uniform (in both branches) given by:

$$\bar{T} = \frac{1-e^{-1/\bar{Q}}}{1+e^{-1/\bar{Q}}} \qquad 0 < s < 1, \tag{19}$$

where the steady state flow rate $\bar{Q}$ is obtained from the solution of the algebraic equation:

$$\bar{Q} = \frac{\alpha}{\varepsilon}\bar{T} = \frac{\alpha}{\varepsilon}\frac{1-e^{-1/\bar{Q}}}{1+e^{-1/\bar{Q}}}. \tag{20}$$

Hence the steady state solution depends on a single parameter α/ε. However, the stability characteristics are strongly affected by the shape of the temperature distribution. As shown in the Appendix, the approximate equations (16) and (17) lead to stable steady state solutions even for the range of parameters that was found to be unstable by Welander [2].

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APPENDIX

Stability characteristics

Welander [2] indicated that there exists a range of parameters for which the steady state motion in the loop (Fig. 1) is unstable. It will be shown here that the approximate equations (16), (17) representing the behavior of the loop do not lead to any instabilities. These equations yield a good approximation for the transient behavior and the exact steady state solution—(19), (20). However, the stability strongly depends on the shape of the temperature distribution and the assumption of a linear profile is not justified for a stability analysis of this loop.

Let us consider small deviations from the steady state in the form:

$$Q(t) = \bar{Q}+Q'(t); \quad T_m(t) = \bar{T}_m + T_m'(t) \tag{A1}$$

where $\bar{T}_m$ and $\bar{Q}$ are given by (19), (20). Introducing (A1) into (16), (17), subtracting the steady state relations and making use of the linearized stability procedure, the following perturbation equations are obtained:

$$(1-m)\frac{dT_m'}{dt} = -2\bar{Q}\left[\frac{mQ'}{\bar{Q}^2}(1+T_m)+(1+m)T_m'\right] \tag{A2}$$

$$\frac{dQ'}{dt} = -\varepsilon Q' + \alpha T_m' \tag{A3}$$

where $m = e^{-1/\bar{Q}}$. The disturbances are taken as:

$$Q' = \hat{Q}\,e^{\sigma t}, \quad T_m' = \hat{T}_m e^{\sigma t} \tag{A4}$$

which, when introduced into (A2),(A3) yield:

$$(1-m)\sigma\hat{T}_m = -2\bar{Q}\left[\frac{m\hat{Q}}{\bar{Q}^2}(1+T_m)+(1+m)\hat{T}_m\right] \tag{A5}$$

$$\sigma\hat{Q} = -\varepsilon\hat{Q}+\alpha\hat{T}_m. \tag{A6}$$

Eliminating $\hat{Q}$ and $\hat{T}_m$ from the last two relations, the characteristic equation for σ is obtained:

$$\sigma^2+\sigma\left(\varepsilon+\frac{2\bar{Q}}{\bar{T}_m}\right)+2\varepsilon\left[\frac{\bar{Q}}{\bar{T}_m}+\frac{2m}{(1-m)^2}\right] = 0. \tag{A7}$$

Since the coefficients in this quadratic equation are always positive, there are no roots for σ with positive real parts and hence no unstable solutions.

COMPORTEMENT TRANSITOIRE DE BOUCLES A CIRCULATION NATURELLE: DEUX BRANCHES VERTICALES AVEC UNE SOURCE THERMIQUE ET UN PUIT PONCTUELS

Résumé—On présente une méthode théorique pour l'évaluation du comportement transitoire de boucles à convection naturelle. La méthode est appliquée à une boucle à deux branches avec une source et un puit de chaleur ponctuels. Le système est représenté par un modèle à une dimension, avec une seule coordonnée spatiale le long de la boucle. Les formes intégrales des équations de quantité de mouvement et d'énergie sont résolues pour obtenir le débit et les températures en fonction du temps. On trouve que cette méthode approchée ne peut reconstruire les caractéristiques de stabilité de la solution exacte en régime permanent.

INSTATIONÄRES VERHALTEN EINES THERMOSYPHON-KREISLAUFS: ZWEI VERTIKALE ZWEIGE MIT PUNKTFÖRMIGER WÄRMEQUELLE UND -SENKE

Zusammenfassung—Es wird eine theoretische Methode zur Beschreibung des instationären Verhaltens eines Kreislaufs mit freier Konvektion angegeben. Die Methode wird auf einen Kreislauf, bestehend aus zwei vertikalen Strecken mit einer punktförmigen Wärmequelle und -senke angewendet. Das System wird durch ein eindimensionales Modell beschrieben, wobei die einzige Raumkoordinate entlang des Kreislaufs verläuft. Durch Ableiten und Lösen der Momenten- und Energiegleichungen in integraler Form erhält man den Durchsatz und die Temperatur als Funktionen der Zeit.

Ee zeigte sich, daß diese Näherungsmethode die Stabilitätskennwerte der exakten stationären Lösung nicht wiedergeben kann.

НЕУСТАНОВИВШИЙСЯ РЕЖИМ РАБОТЫ ЕСТЕСТВЕННЫХ ЦИРКУЛЯЦИОННЫХ КОНТУРОВ: ДВА ВЕРТИКАЛЬНЫХ ПАТРУБКА С ТОЧЕЧНЫМИ ИСТОЧНИКАМИ И СТОКАМИ ТЕПЛА

Аннотация — Предложен теоретический метод оценки неустановившегося режима работы контуров со свободной конвекцией. Метод применим к контуру, состоящему из двух вертикальных патрубков с точечными источниками и стоками тепла. Система представлена одномерной моделью с единственной пространственной координатой, направленной вдоль контура. Для получения временной зависимости скорости и температуры потока выведены и решены интегральные уравнения количества движения и энергии. Найдено, что с помощью предложенного приближенного метода нельзя получить характеристики устойчивости точного стационарного решения.